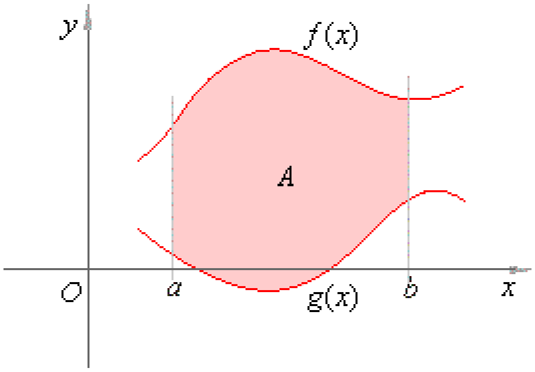
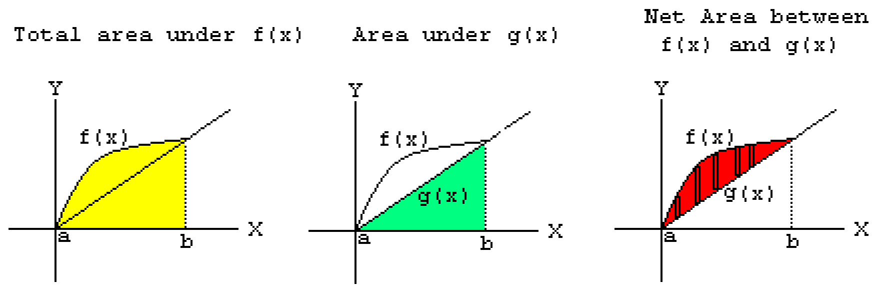
6.1 Area Between Curves

Consider the region **A** that lies between two curves, ***y = f(x)*** and ***y = g(x)*** and the vertical lines **x = a** and **x = b**, where **f** and **g** are continuous functions and ***f(x) ≥ g(x)*** for all **x** in [a, b].





The area **A** of the region bounded by the curves ***y = f(x)*** , ***y = g(x),*** and the lines **x = a**, **x = b**, where ***f and g*** are continuous and ***f(x) ≥ g(x)*** for all **x** in [a, b] is

$$\boldsymbol{A = \int_a^b [f(x) - g(x)]dx}$$

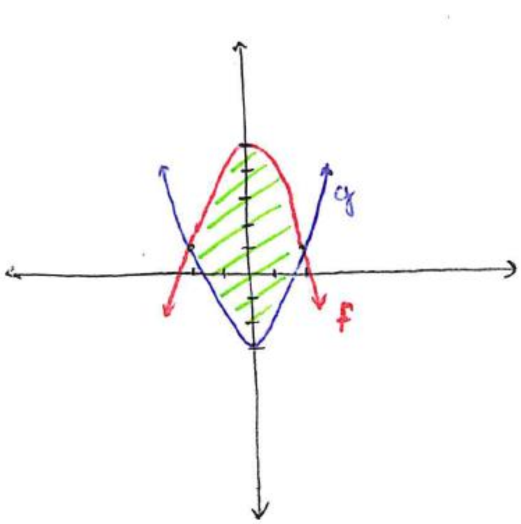
Example: Find the area of the region bounded by the graphs $f(x) = 5 - x^2$ and $g(x) = x^2 - 3$.

This problem has multiple steps. We need to graph **f(x)** and **g(x)** to determine which function is greater in the domain. In addition, we need to solve for the domain. Lets first find the intersection by setting the functions equal to each other.

$$5 - x^2 = x^2 - 3, \text{ solve for x}$$

$$8 = 2x^2 \Rightarrow 4 = x^2 \Rightarrow \boldsymbol{x = \pm 2}$$

$\boldsymbol{x = \pm 2}$ become the limits of integration. Below is the graph of the two functions.



Note that **f(x) ≥ g(x)** ∀ **x** in [-2, 2] (∀ *means "for all"*)

Therefore the region has an area of:

$A = \int_{-2}^{2}[(5 - x^2) - (x^2 - 3)]dx$ since these functions are even and the area has y-axis symmetry, then $A = 2\int_0^2 (8 - 2x^2)dx = 2\left(8x - \frac{2}{3}x^3\right)\Big]_0^2 = \boldsymbol{\frac{64}{3}}$

Example: Find the area bounded by the graphs of $f(x) = -x^2 + 3x + 6$ *and* $g(x) = |2x|$.

This is a compound problem because of the absolute value, but we know that

$$g(x) = |2x| = \begin{cases} 2x & if\ x \geq 0 \\ -2x & if\ x < 0 \end{cases}$$

To find the intersections, we need to set **f(x) = g(x)** and solve for **x**. We have to do this in two parts.

For **x < 0:**

$-x^2 + 3x + 6 = -2x$

$-x^2 + 5x + 6 = 0$

$(x + 1)(x - 6) = 0$

$x = -1 \;\&\; x = 6$

For **x ≥ 0:**

$-x^2 + 3x + 6 = 2x$

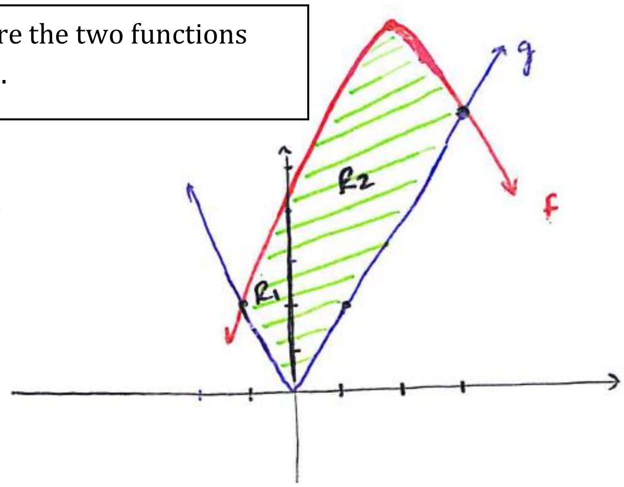
$-x^2 + x + 6 = 0$

$(x + 2)(x - 3) = 0$

$x = -2 \;\&\; x = 3$

The area between the curves lies between the intersection points of [-1, 3], we do not use 6 & -2.

Below are the two functions graphed.



Since we have a piecewise function, we need to separate the problem into two integrals (two regions).

$$A = \int_{-1}^{0} [(-x^2 + 3x + 6) - (-2x)]dx + \int_{0}^{3} [(-x^2 + 3x + 6) - (2x)]\,dx$$

$$A = \int_{-1}^{0} (-x^2 + 5x + 6)dx + \int_{0}^{3} (-x^2 + x + 6)dx$$

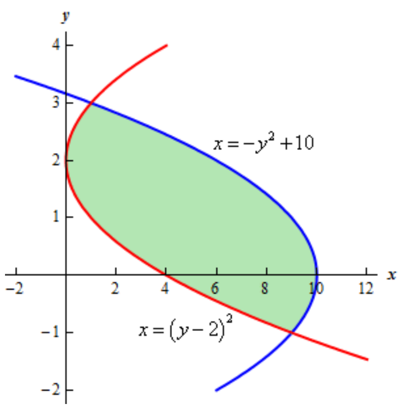
$$A = \left(-\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x\right)\Bigg]_{-1}^{0} + \left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\right)\Bigg]_{0}^{3}$$

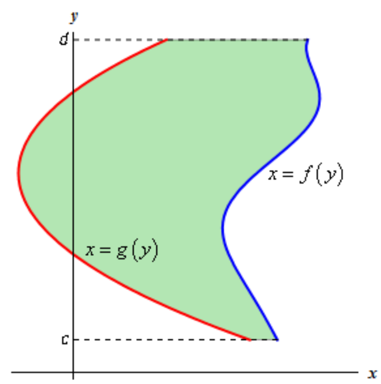
$$A = \left[0 - \left(\frac{1}{3} + \frac{5}{2} - 6\right)\right] + \left[\left(-9 + \frac{9}{2} + 18\right) - 0\right] = \mathbf{\frac{50}{3}}$$

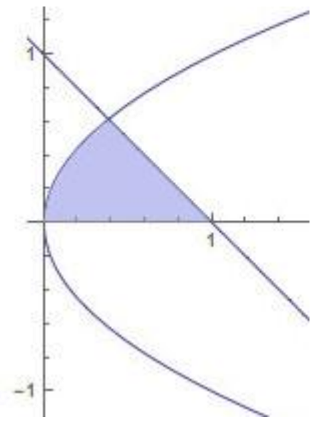
There are times where, if it is convenient (or possibly necessary), to reverse the roles of **x** and **y.** If the region is bounded by courves with equations $x = f(y)$, $x = g(y)$, $y = c$, & $y = d$ where **f** and **g** are continuous and $f(y) \geq g(y)$ for ∀ in [c, d], then the area is

$$A = \int_c^d [f(y) - g(y)]dy$$

Consider the following curves. (Notice these curves may not be functions.)

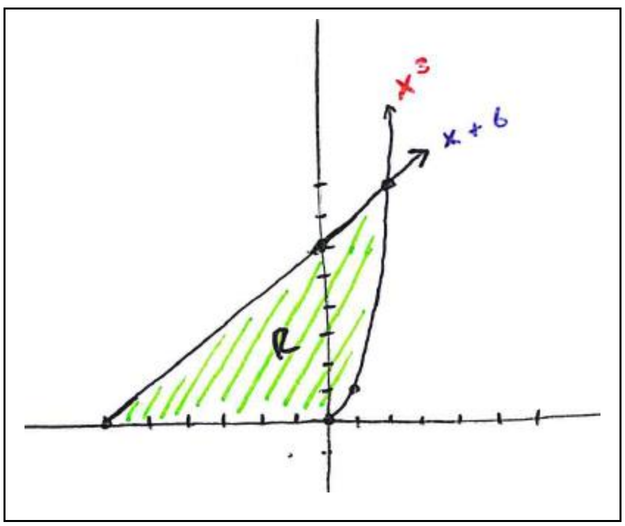






Example: Find the area of the region **R** bounded by the graphs of $y = x^3$, $y = x + 6$, and the $x - axis$.

Graph the functions.



Find the point(s) of intersection for **y**.

Since $y = x^3$, we can rewrite it as $x = y^{\frac{1}{3}}$, and for the line

$y = x + 6$, we can rewrite it as $x = y - 6$. So $y^{\frac{1}{3}} = y - 6 \Rightarrow$

$y = (y - 6)^3$. Expand and solve for y. (Precalculus!!!)

(Hint: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$)

$y = y^3 - 18y^2 + 108y - 216, 0 = y^3 - 18y^2 + 107y - 216$

Using the $\pm\frac{P}{q}$ method of finding zeros we see that y = 8 is the only real solution.

Notice that the region goes from **y = 0** to **y = 8**. These are the limits of integration.

$\int_0^8 \left[\left(y^{\frac{1}{3}}\right) - (y - 6)\right] dy = \int_0^8 \left(y^{\frac{1}{3}} - y + 6\right) dy = \left(\frac{3}{4}y^{\frac{4}{3}} - \frac{1}{2}y^2 + 6y\right)\Big|_0^8 = \left(\frac{3}{4} \cdot 16 - 32 + 48\right) - 0 = \mathbf{28}$

The chart below summarizes the method of finding the area between two or more curves.